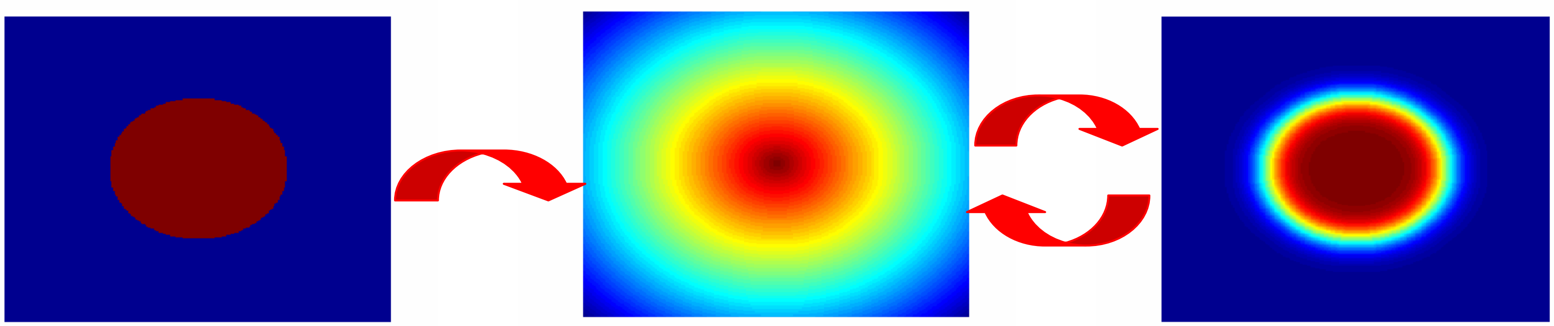
A Shape Representation based on the Logarithm of Odds

By

Kilian Pohl, John Fisher, William Wells



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Overview

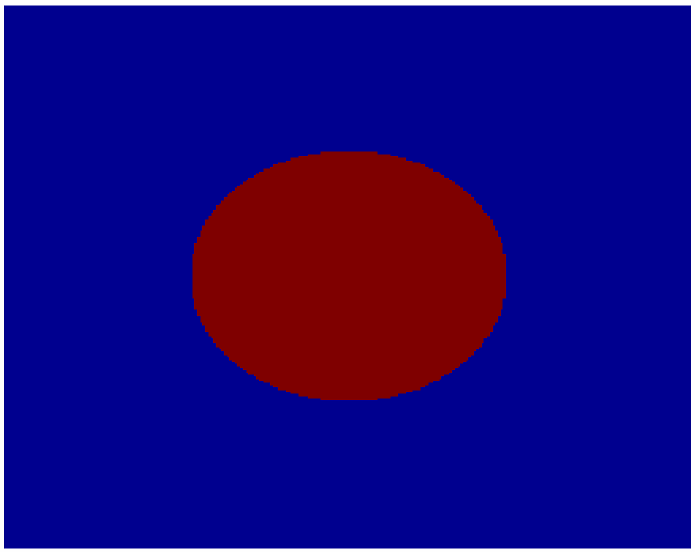
Motivation

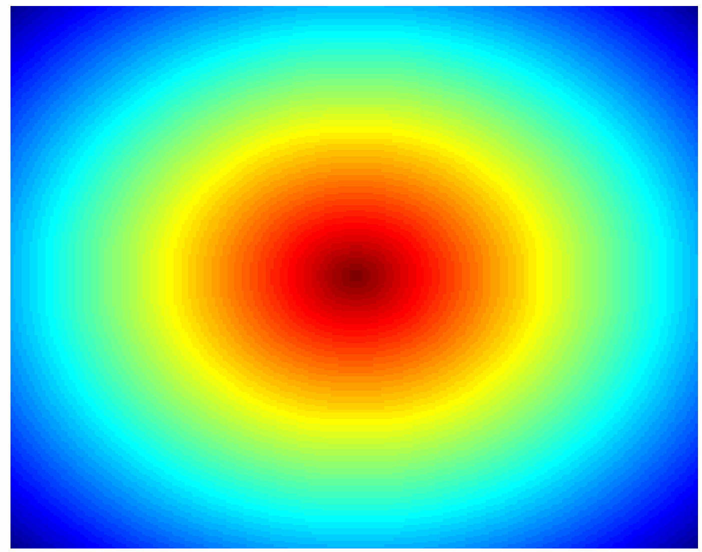
LogOdds and Its Properties

Experiment

Additional Applications

Signed Distance Maps





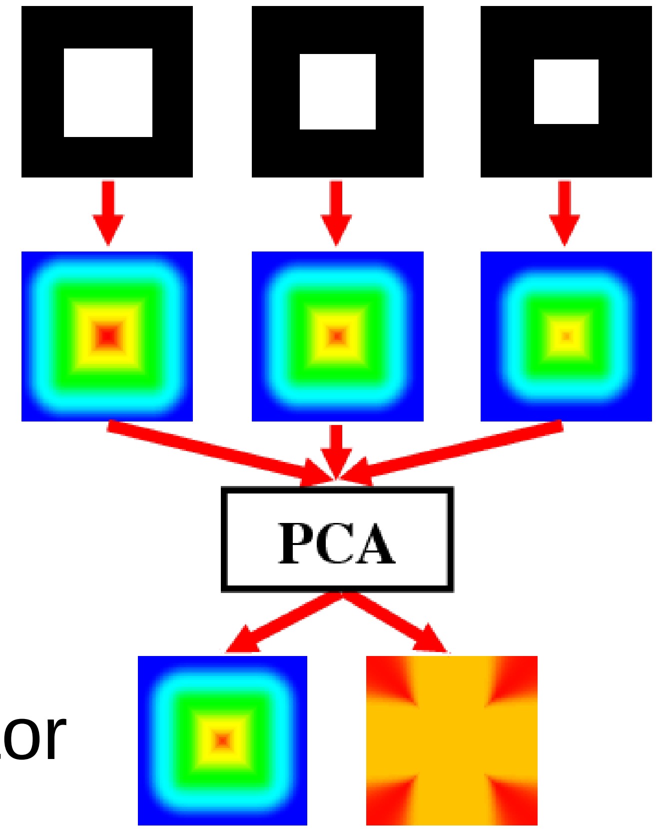
Outside  Inside

Principle Component Analysis

Labelmap

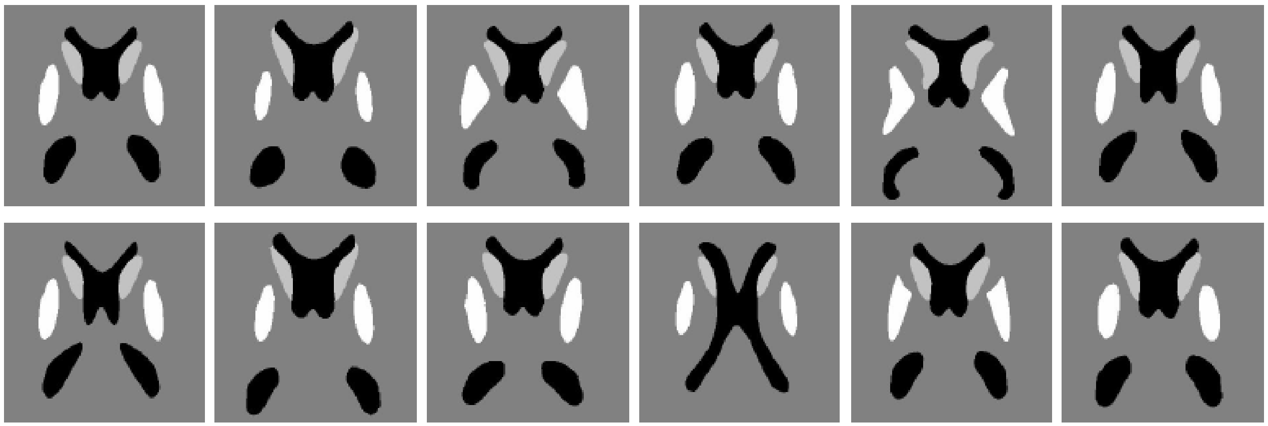
Distance map

Mean & Eigenvector



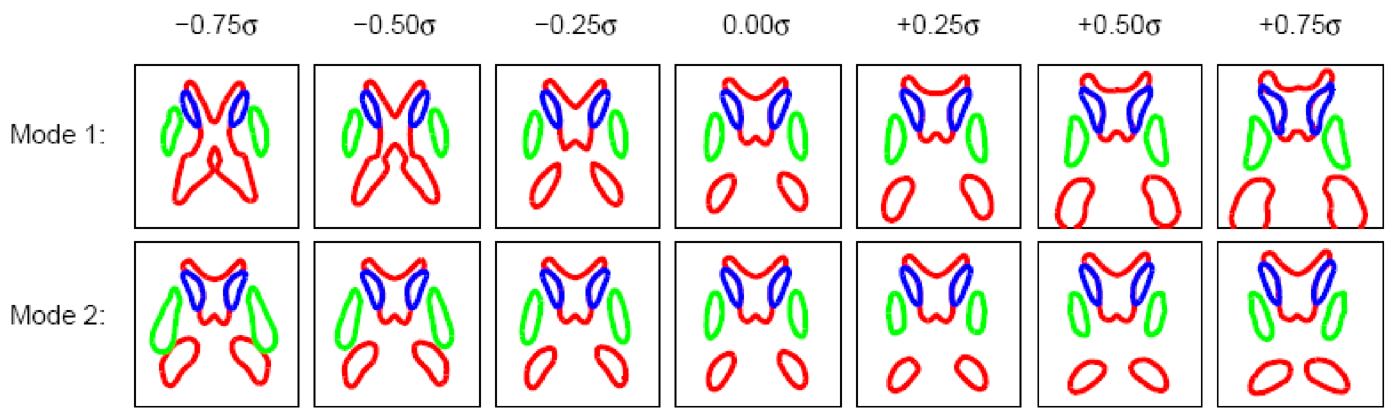
Leventon et al.: “Statistical Shape Influence in Geodesic Active Contours”, Conf. on Computer Vision and Pattern Recognition, 2000

PCA for Multiple Objects



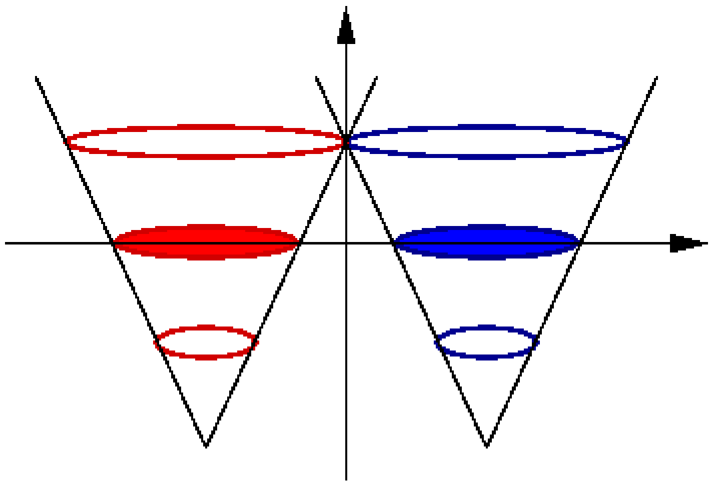
Data

PCA

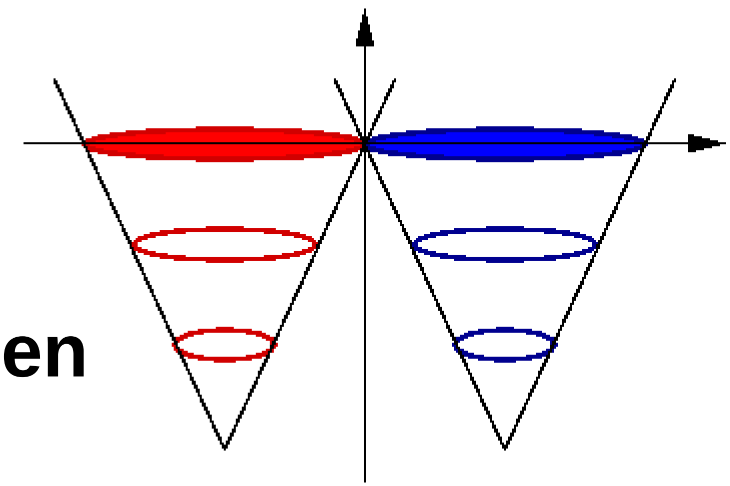


Tsai et al.: “Mutual Information in Coupled Multi-Shape Model for Medical Image Segmentation”, Medical Image Analysis, 2004

Example: PCA of Two Circles



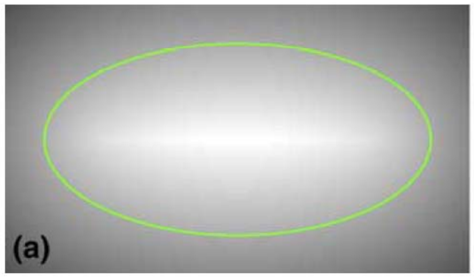
Captures the covariation between two objects



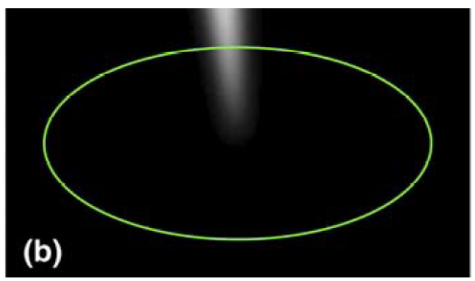
Problem:
Define boundary between overlapping shapes?

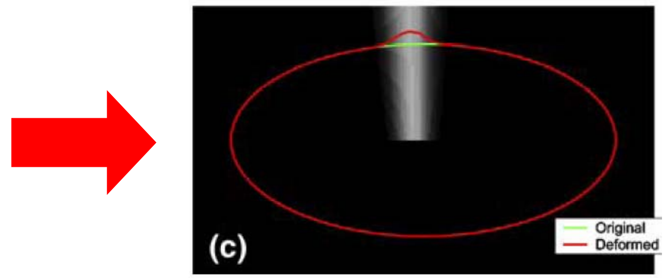
Result of PCA

Mean



Eigenvector





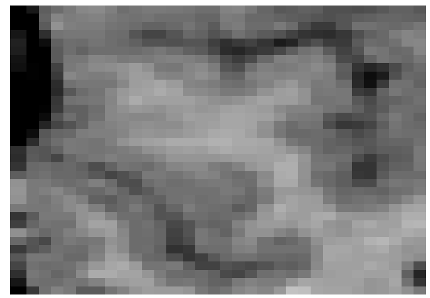
Resulting Object

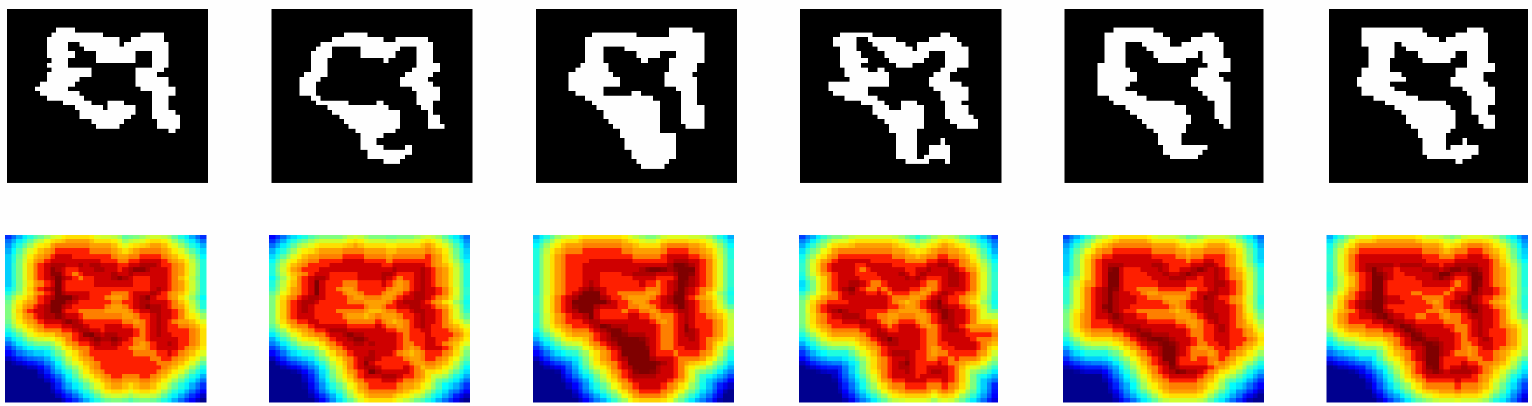
Problem:

Does generally not result in distance map

Golland et al.: “Detection and Analysis of Statistical Differences in Anatomical Shape”, Medical Image Analysis, 2004

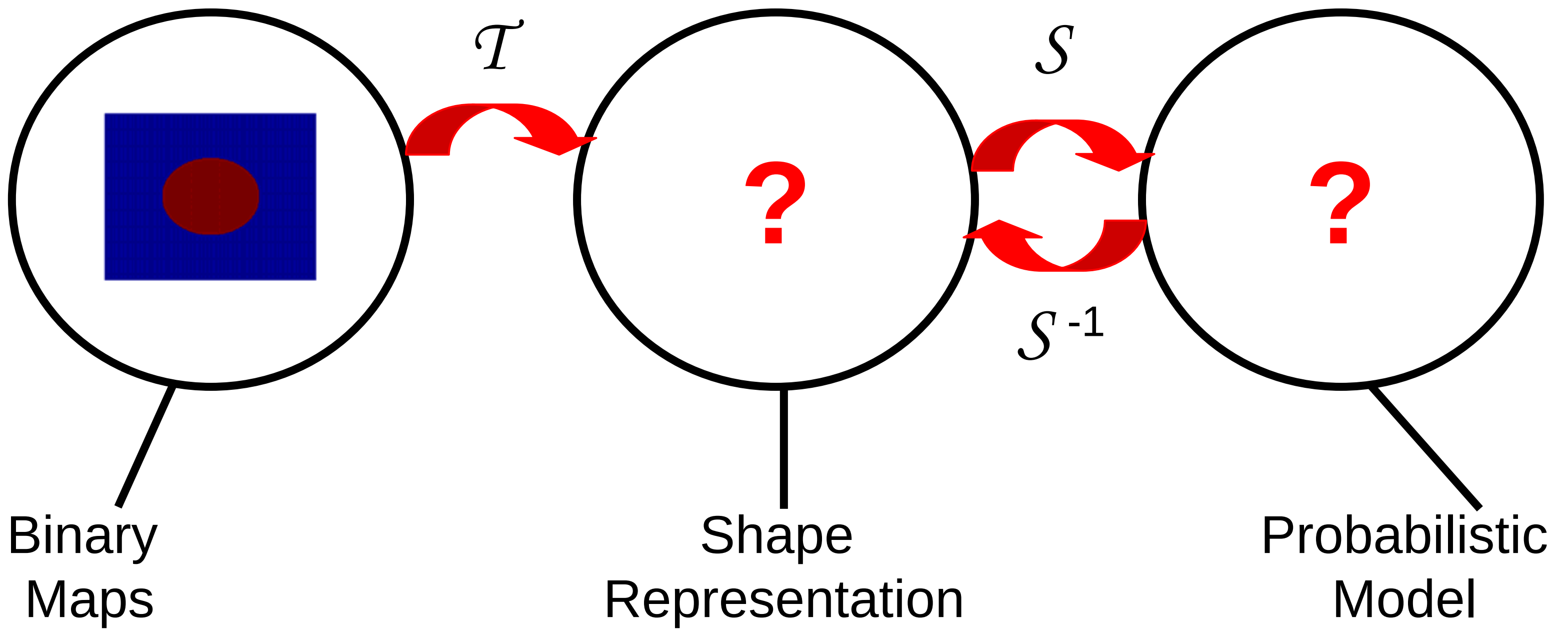
Indistinct Boundaries





Problem:
Cannot capture uncertainty of shape

Goal



Define Shape Representation, that

- alternative transformation $\mathcal{T}$ to distance maps
- defines a linear vector space and maintains intrinsic properties
- relates to a probabilistic model via $\mathcal{S}$ indicating certainty about boundary location

Overview

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LogOdds and Its Properties

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Probabilistic Interpretation

Experiment

Additional Applications

The Logarithm of Odds

Definition:

The **LogOdds** of a probability $p \in [0,1]$ is defined as the logarithm of the odds: the ratio of the probability *p* and its complement *1- p*

$$logit(p) \triangleq \log\left(\frac{p}{1-p}\right) = \log p - \log(1-p)$$

The inverse of the log odds function *logit(·)* is the standard logistic function or Sigmoid function

$$\mathcal{P}(t) \triangleq \frac{1}{1+e^{-t}}$$

The Space of LogOdds

Definition:

The LogOdds space is composed by the LogOdds of all probabilities :

$$\mathbb{L} \triangleq \{\, logit(p) \mid p \in \mathbb{P} \,\}$$

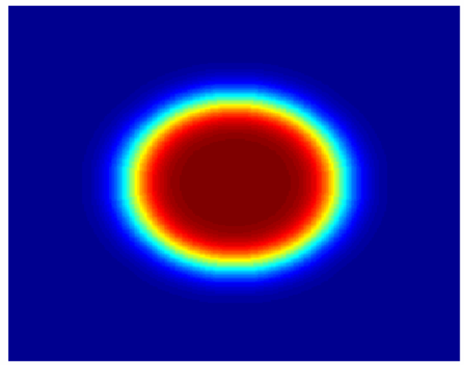
where

$$\mathbb{P} \triangleq \{\, p \mid p \text{ is a probability} \,\}$$

represents the space of probabilities.

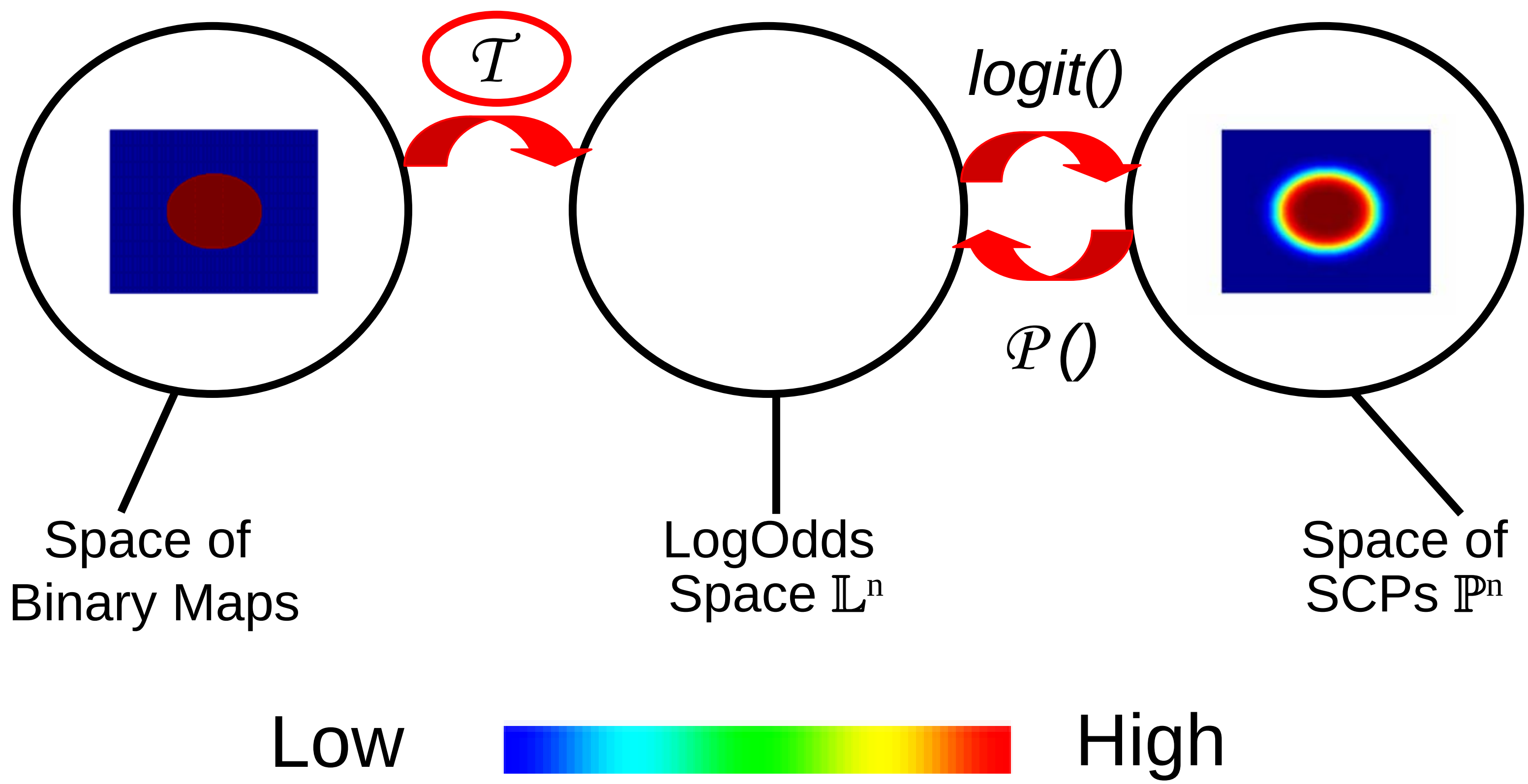
$\mathbb{L}$ is equivalent to $\mathbb{R} \Rightarrow \mathbb{L}^n = \mathbb{R}^n$ is a **vector space**

Example of $\mathbb{P}^n$

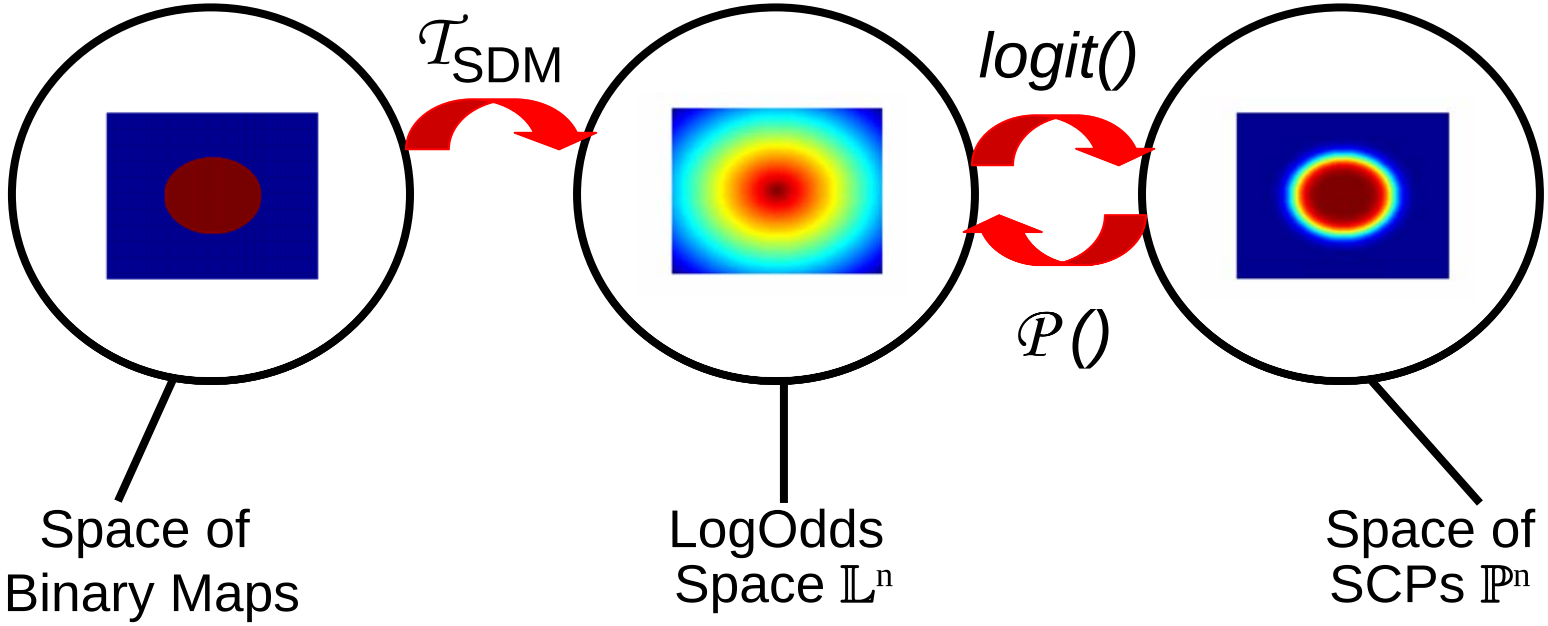


If voxels are Independent-Identical-Distributed (IID) then any element of $\mathbb{P}^n$ defines a Space-Conditioned Probability (SCP), which is the probability that an object is present at a given the voxel location

LogOdds Space



Example for $\mathcal{T}$ - SDM



$\mathcal{T}_{SDM}$ = Any monotonic transform of the Signed Distance Map (SDM) is in $\mathbb{L}^n$

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Revisit Multi Rater Example

Defining an Abelian Group in $\mathbb{P}$

Definition:

The *probabilistic addition,* $\oplus$, of p_1, $p_2 \in \mathbb{P}$ is defined as

$$p_1 \oplus p_2 \triangleq \mathcal{P}(logit(p_1) + logit(p_2)) = \frac{p_1 \cdot p_2}{p_1 \cdot p_2 + (1 - p_1)(1 - p_2)}$$

Properties:

- $(\mathbb{P}, \oplus)$ defines an Abelian group with the null element 0.5 and the additive inverse (1-p)
- $\oplus$ corresponds to normalized multiplication of two probabilities
- The complement

$$1 - (p_1 \oplus p_2) = (1 - p_1) \oplus (1 - p_2)$$

$\oplus$ and Bayes' Rule

Let the normalized likelihood for an event *A* with respect to the random variable *X* be

$$p_1 \triangleq \frac{P(A|X)}{P(A|X)+P(A|\bar{X})} = 1-\frac{P(A|\bar{X})}{P(A|X)+P(A|\bar{X})}$$

along with

$$p_2 \triangleq P(X) = 1 - P(\bar{X})$$

then

$$p_1 \oplus p_2 = \frac{p_1 \cdot p_2}{p_1 \cdot p_2 + (1-p_1)(1-p_2)}$$

$$= \frac{\frac{P(A|X)}{P(A|X)+P\left(A|\bar{X}\right)} P\left(X\right)}{\frac{P(A|X)}{P(A|X)+P\left(A|\bar{X}\right)} P\left(X\right) + \frac{P\left(A|\bar{X}\right)}{P(A|X)+P\left(A|\bar{X}\right)} P\left(\bar{X}\right)}$$

$$= \frac{P\left(A|X\right) P\left(X\right)}{P\left(A\right)} = P(X|A)$$

Defining a Vector Space in $\mathbb{P}$

Definition:

The *probabilistic scalar multiplication,* $\circledast$*,* between the scalar $\alpha \in \mathbb{R}$ and probability $p \in \mathbb{P}$ is defined as

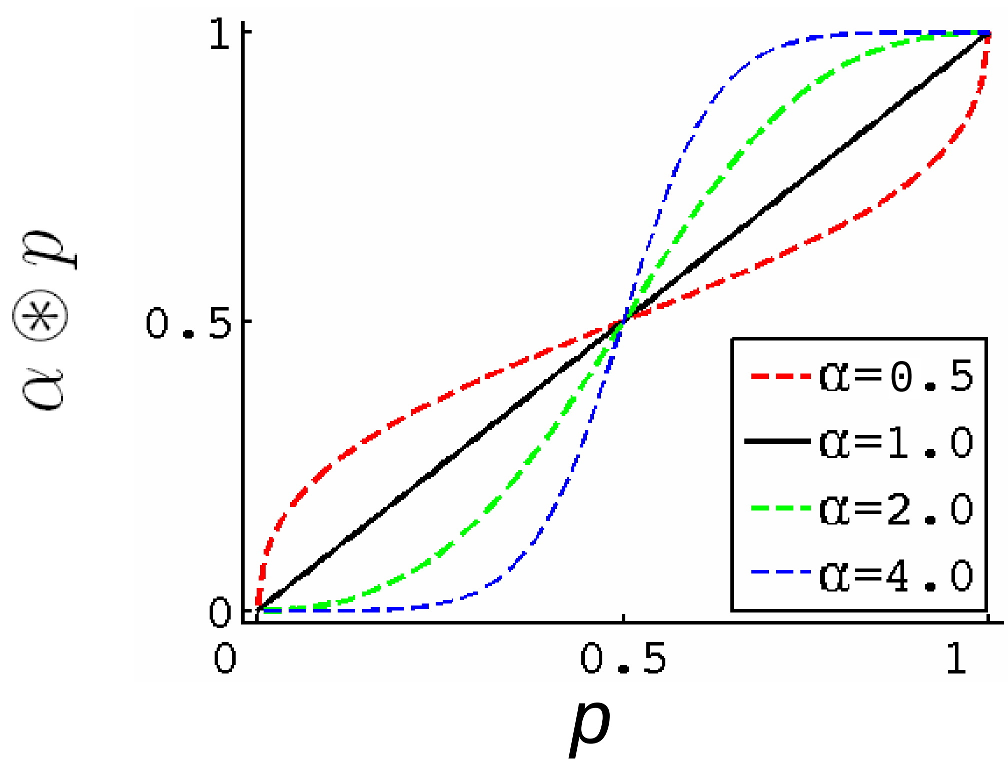
$$\alpha \circledast p \triangleq \mathcal{P}(\alpha * logit(p)) = \frac{1}{1 + e^{-\alpha \cdot \log(\frac{p}{1-p})}} = \frac{p^\alpha}{p^\alpha + (1-p)^\alpha}$$

Properties:

- $(\mathbb{P}, \oplus, \circledast)$ defines a Vector space with 1 as the identity of the scalar multiplication
- $(\mathbb{P}, \oplus, \circledast)$ is equivalent to $(\mathbb{L}, +, *)$
- The complement

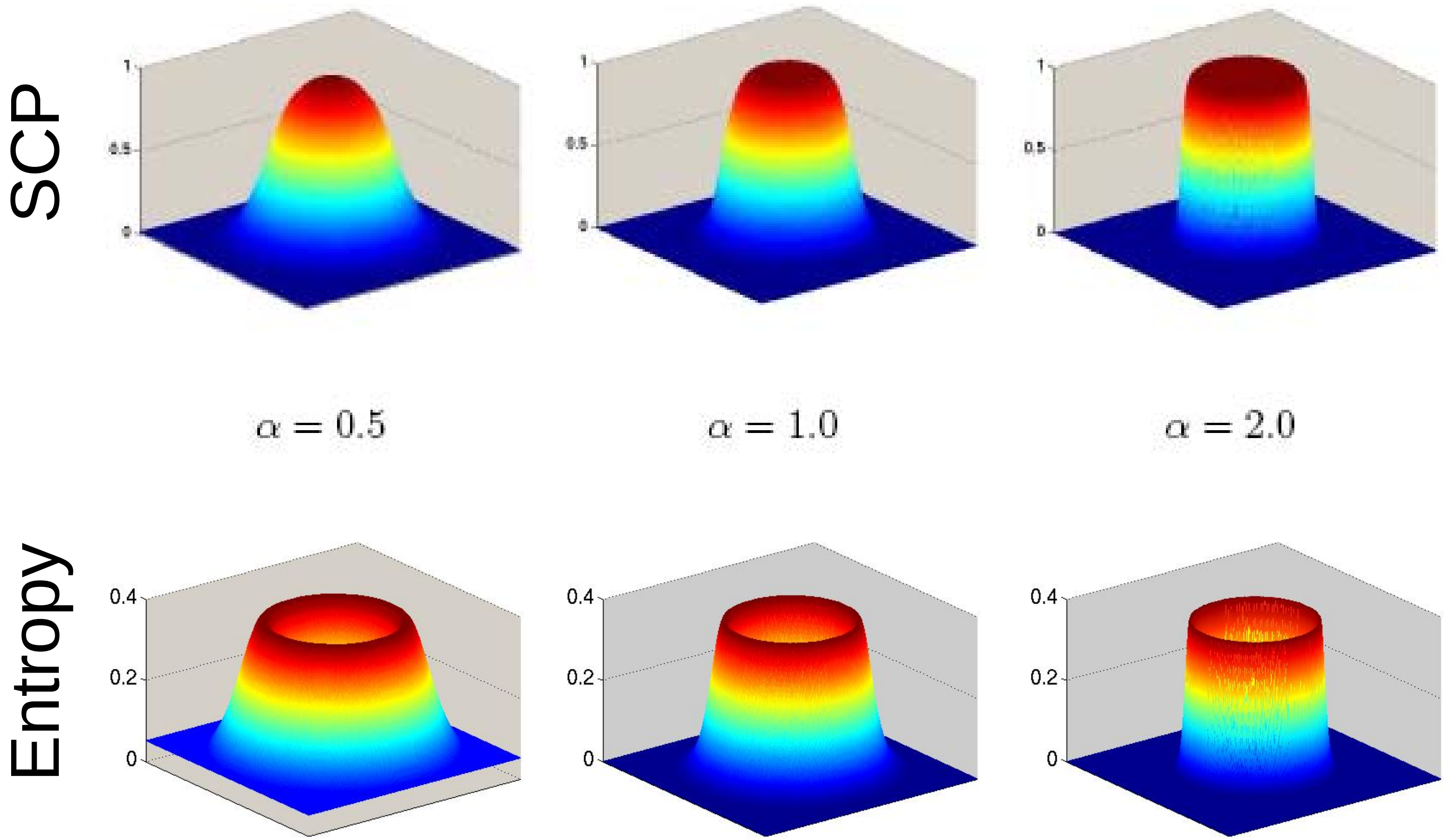
$$1 - (\alpha \circledast p) = \alpha \circledast (1-p) = -\alpha \circledast p.$$

Impact of α



If voxels are iid then α represents the certainty within the boundary location of a binary image.

Scalar Multiplication of an SCP

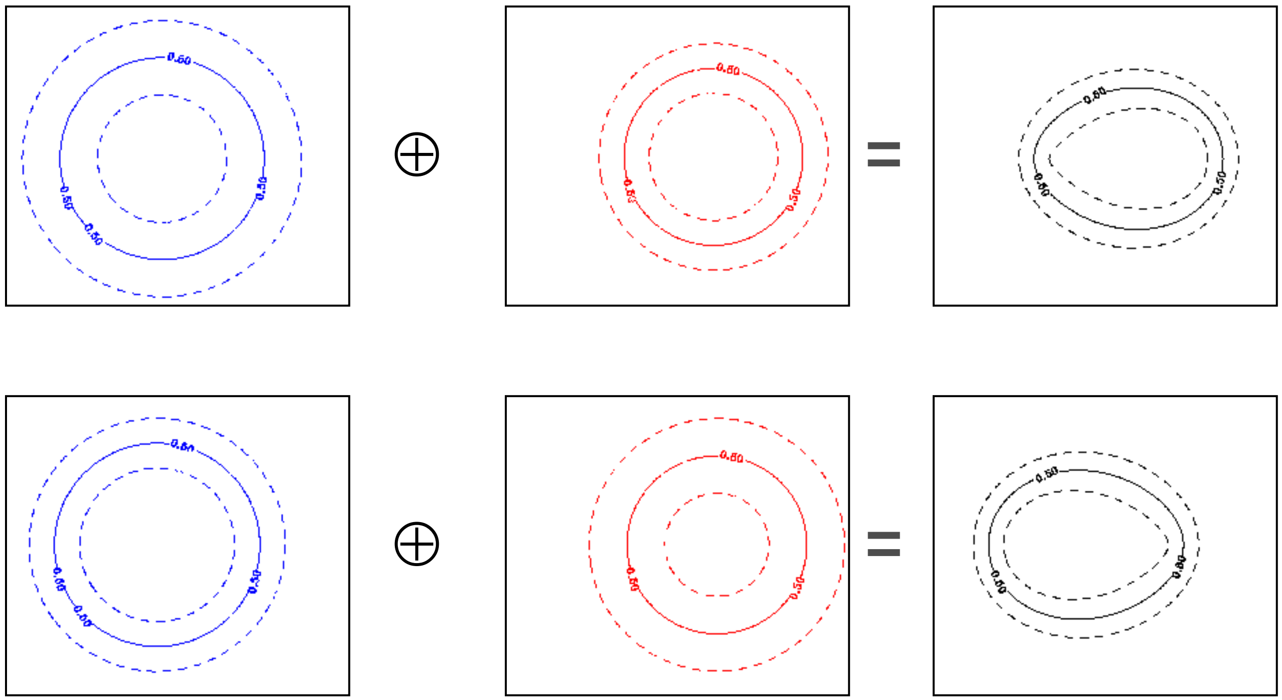


Addition and Multiplication in $\mathbb{P}^n$

$\alpha_{left} \circledast SCP_{left} \oplus \alpha_{right} \circledast SCP_{right} = SCP_{re}$

$\alpha_{left} < \alpha_{right}$

$\alpha_{left} > \alpha_{right}$



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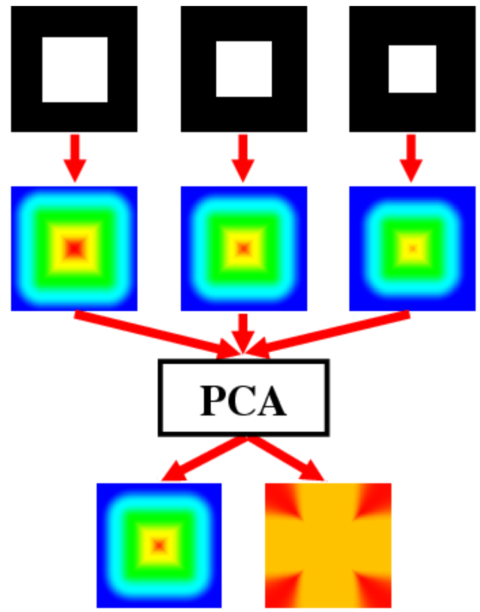
LabelMap

LogOdds

Mean (M) &
Matrix of Eigenvectors (E)

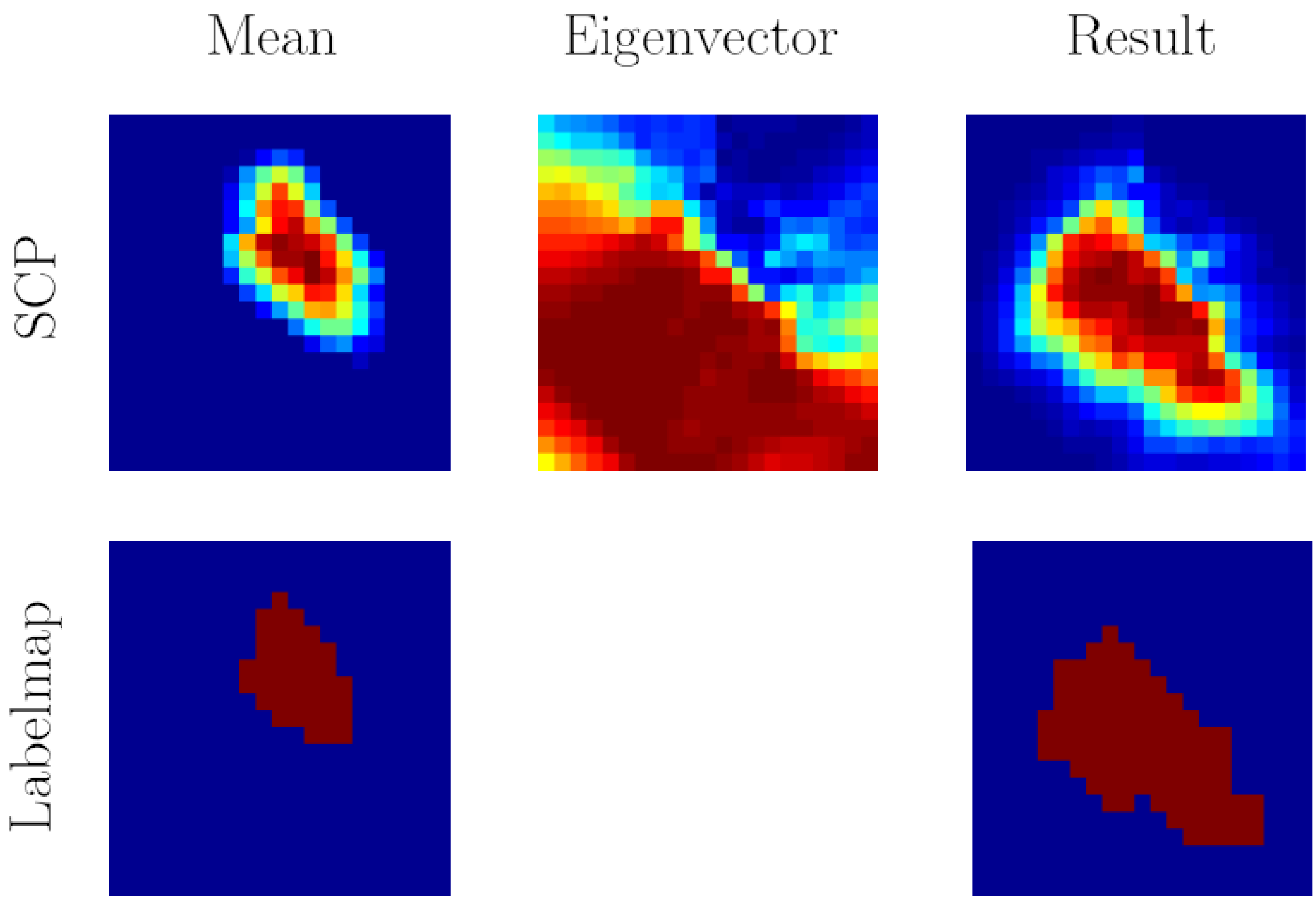


where LogOdds $V = M + \alpha E \in \mathbb{L}^{n \times m}$ with

n = number of objects without the background

m = number of voxels in the image

Example



Define Segmentation Model

Data

Labelmap T

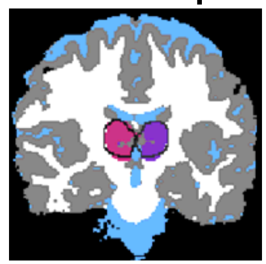
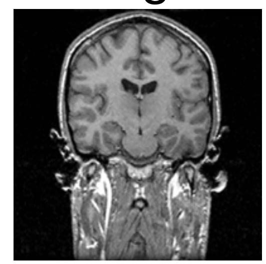
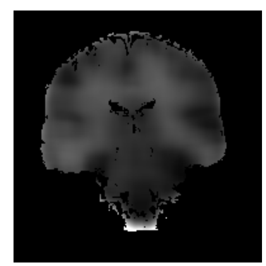


Image I



$$\hat{\mathcal{B}} = \arg\max_{\mathcal{B}} \log\left(\sum_{\mathcal{T}} \mathrm{P}(\mathcal{T}, \mathcal{B} | \mathcal{I})\right)$$

Parameter



Inhomogeneity B

Define Segmentation Model

Data

Labelmap T

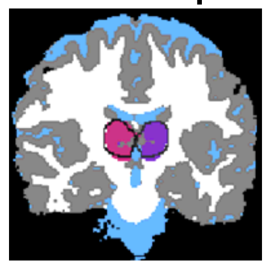
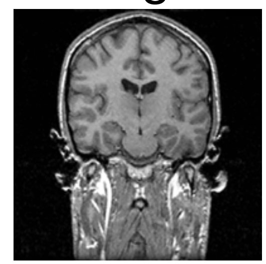
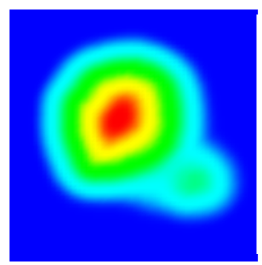


Image I

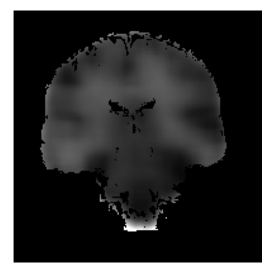


$$(\hat{\theta}, \hat{\mathcal{B}}) = \arg\max_{\theta, \mathcal{B}} \log\left(\sum_{\mathcal{T}} \mathrm{P}(\mathcal{T}, \theta, \mathcal{B} | \mathcal{I})\right)$$

Parameter



Shape θ



Inhomogeneity B

EM Implementation

Expectation Step: Calculate **Weights**

$$\mathcal{W}_{\mathrm{x}} \equiv \mathrm{E}_{\mathcal{T}|\mathcal{I},\mathcal{B}',\theta'}(\mathcal{T}_{\mathrm{x}})$$

Maximization Step: Optimize the **Estimates**

$$\mathcal{B}' \leftarrow \arg\max_{\mathcal{B}} \sum_{\mathrm{x}} \mathcal{W}_{\mathrm{x}}^{\mathrm{t}} \log \mathrm{P}(\mathcal{I}_{\mathrm{x}}|\mathcal{T}_{\mathrm{x}}, \mathcal{B}_{\mathrm{x}}) + \log \mathrm{P}(\mathcal{B})$$

$$\theta' \leftarrow \arg\max_{\theta} \sum_{\mathrm{x}} \mathcal{W}_{\mathrm{x}}^{\mathrm{t}} \log \mathrm{P}(\mathcal{T}_{\mathrm{x}}|\theta) + \log \mathrm{P}(\theta)$$

Defining Likelihood of Shape

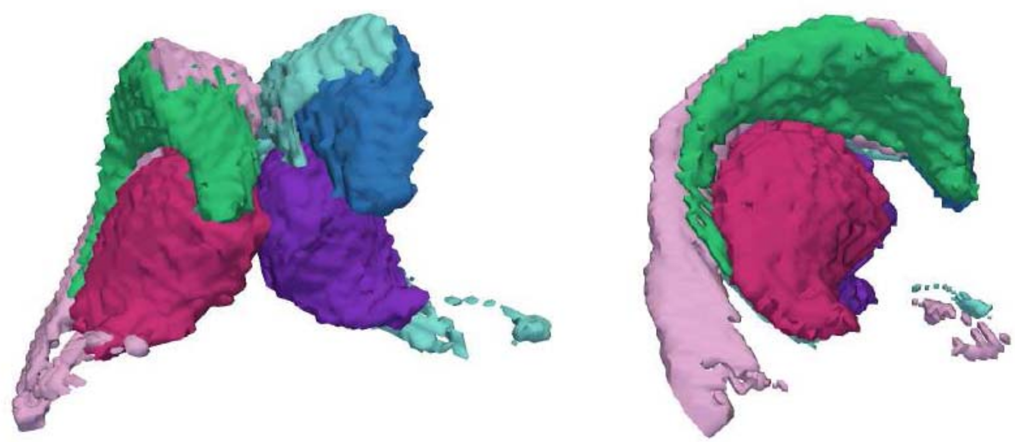
Level Set Formulation

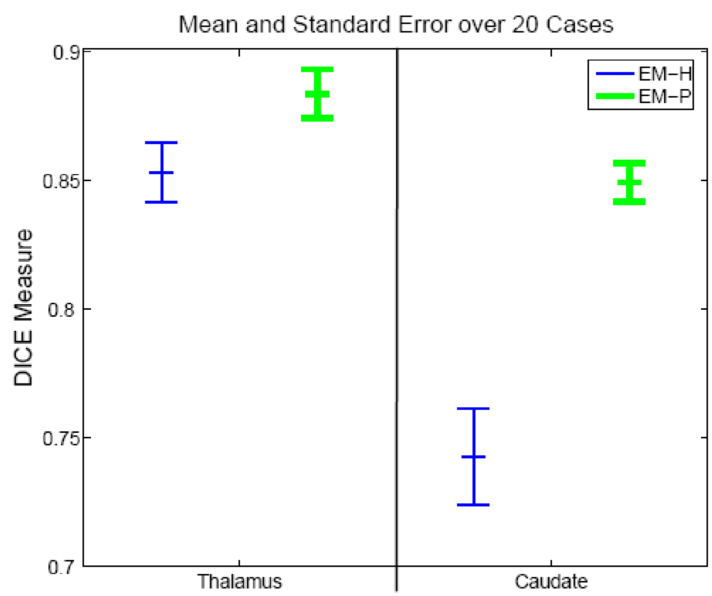
$$\mathcal{H}(v) := \begin{cases} 1 & , \text{ if } v \geq 0 \\ 0 & , \text{ otherwise} \end{cases} \Rightarrow P_{\mathcal{H}}(\mathcal{T}_x = e_a|\theta) \triangleq \frac{\mathcal{H}(\mathcal{D}_{\theta,a}(x))}{\sum_{a'} \mathcal{H}(\mathcal{D}_{\theta,a'}(x))}$$

Log Odds Representation

$$P_{\mathcal{P}}(\mathcal{T}_x = e_a|\theta) \triangleq [\mathcal{P}_M(\mathcal{D}_\theta(x))]_a = \frac{e^{\mathcal{D}_{\theta,a}(x)}}{1 + \sum_{a'=1,\ldots,M-1} \; e^{\mathcal{D}_{\theta,a'}(x)}}$$

Study of 20 Cases





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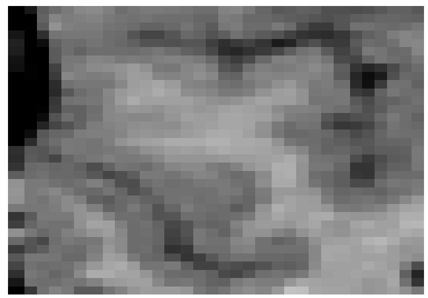
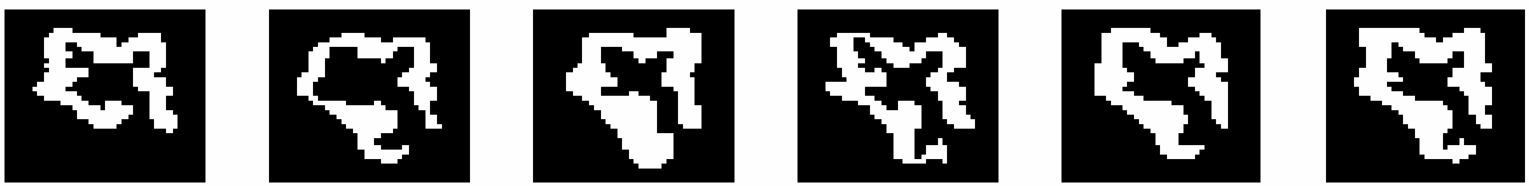
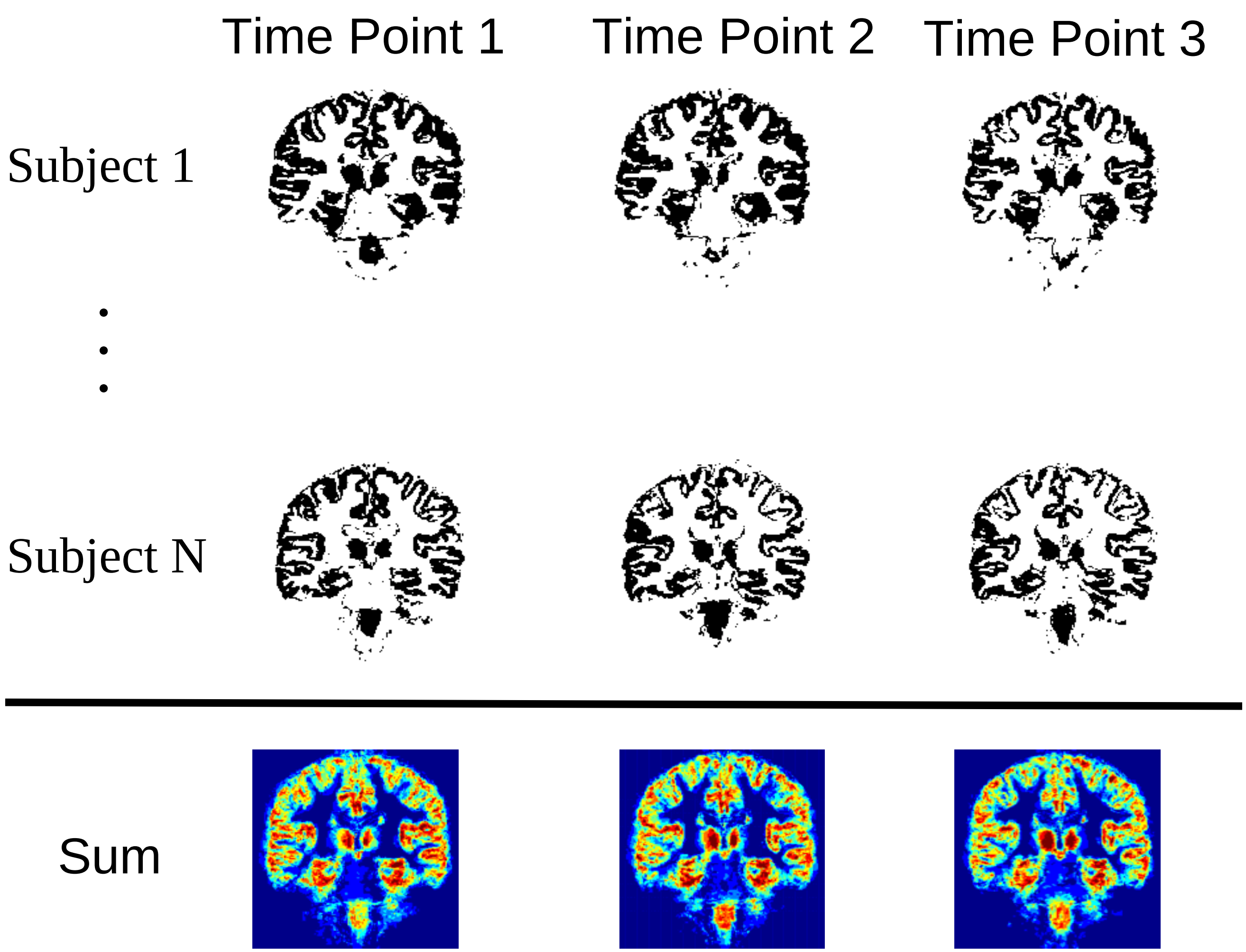
Multi Rater Example Revisited

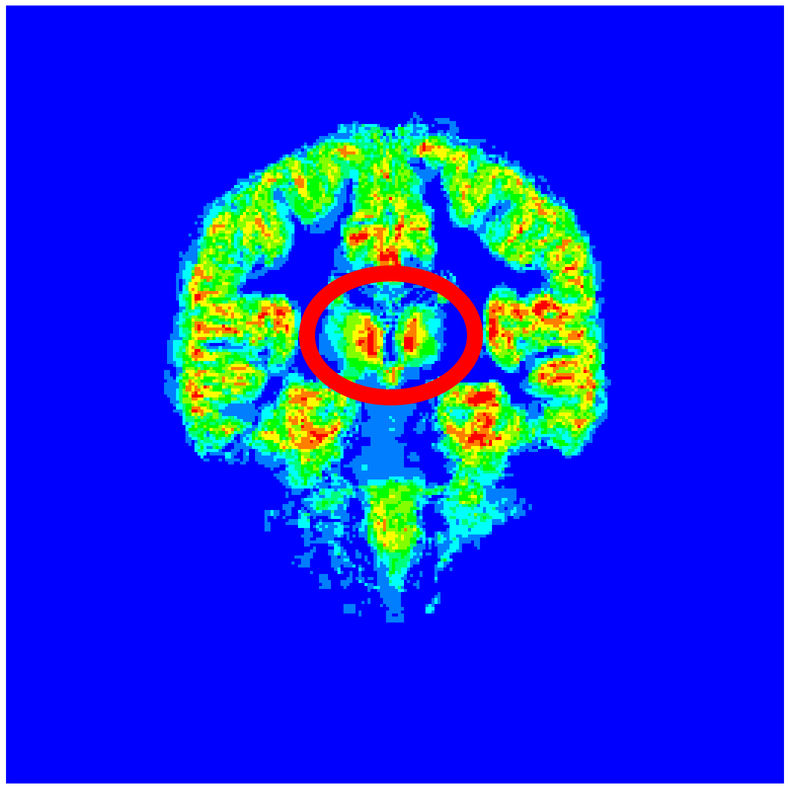




Longitudinal Study

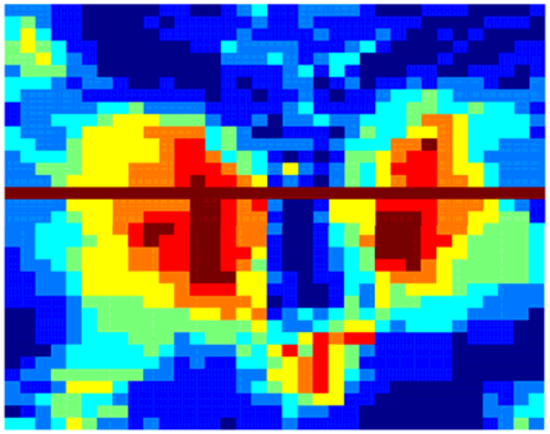


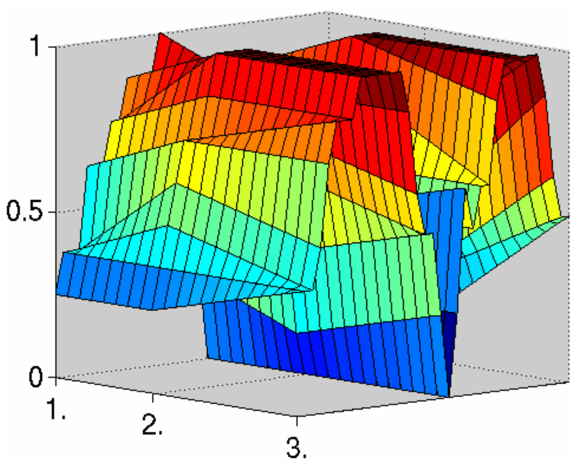
Interpolation



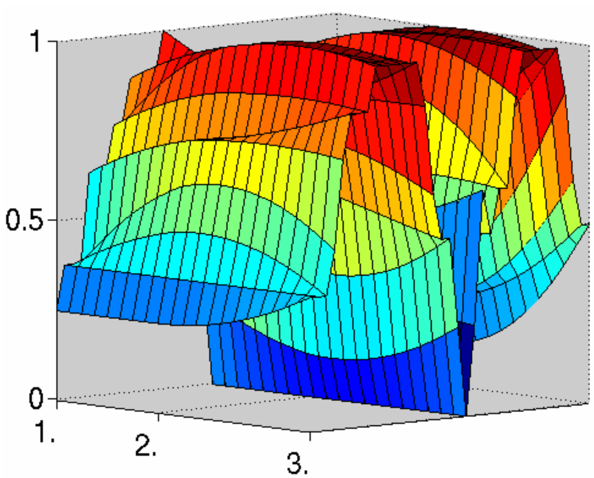
Low High

Convex vs. Non Convex





Linear Convex Combination



Quadratic Spline Interpolation

Summary

We presented a new shape representation called LogOdds. The representation

- encodes shapes as well as their variations
- defines a linear vector space
- provides a spatial probabilistic interpretation
- addresses certain problems in vision
- achieves higher accuracy then the level-set representation in the experiment.

Thank You

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